Exercise 19

A series circuit contains a resistor with $R = 40\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.0025$ F, and a 12-V battery. The initial charge is $Q = 0.01$ C and the initial current is 0. Find the charge at time t.

Solution

The initial value problem for charge in this LRC series circuit with a V-volt battery is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V, \quad Q(0) = 0.01, \quad Q'(0) = 0.$$

Because the ODE is linear and inhomogeneous, the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$Q = Q_c + Q_p$$

The complementary solution satisfies the associated homogeneous equation.

$$L\frac{d^2Q_c}{dt^2} + R\frac{dQ_c}{dt} + \frac{Q_c}{C} = 0 \tag{1}$$

Because it has constant coefficients, it has solutions of the form $Q_c = e^{rt}$.

$$Q_c = e^{rt} \quad \rightarrow \quad \frac{dQ_c}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2Q_c}{dt^2} = r^2e^{rt}$$

Substitute these formulas into equation (1).

$$L(r^2e^{rt}) + R(re^{rt}) + \frac{1}{C}(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$Lr^2 + Rr + \frac{1}{C} = 0$$

Solve for r, noting that with the given values $4LC - R^2C^2 > 0$.

$$LCr^2 + RCr + 1 = 0$$

$$r = \frac{-RC \pm \sqrt{R^2C^2 - 4(LC)(1)}}{2(LC)} = \frac{-RC \pm i\sqrt{4LC - R^2C^2}}{2LC}$$

$$r = \left\{\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}, \frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}\right\}$$

Two solutions to equation (1) are

$$\exp\left(\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}t\right) \quad \text{and} \quad \exp\left(\frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}t\right).$$

According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
Q_c(t) &= C_1 \exp\left(\frac{-RC - i\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(\frac{-RC + i\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&= C_1\left(-\frac{R}{2L}t\right)\exp\left(-\frac{i\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(-\frac{R}{2L}t\right)\exp\left(\frac{i\sqrt{4LC - R^2C^2}}{2LC}t\right) \\
&= \exp\left(-\frac{R}{2L}t\right)\left[C_1 \exp\left(-\frac{i\sqrt{4LC - R^2C^2}}{2LC}t\right) + C_2 \exp\left(\frac{i\sqrt{4LC - R^2C^2}}{2LC}t\right)\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left[C_1\left(\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t - i\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)\right. \\
&\qquad\qquad\left. + C_2\left(\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + i\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left[(C_1 + C_2)\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + (-iC_1 + iC_2)\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right] \\
&= \exp\left(-\frac{R}{2L}t\right)\left(C_3\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right)
\end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$L\frac{d^2Q_p}{dt^2} + R\frac{dQ_p}{dt} + \frac{Q_p}{C} = V \tag{2}$$

Since the inhomogeneous term is a polynomial of degree 0, the particular solution is $Q_p = A$.

$$Q_p = A \quad \rightarrow \quad \frac{dQ_p}{dt} = 0 \quad \rightarrow \quad \frac{d^2Q_p}{dt^2} = 0$$

Substitute these formulas into equation (2).

$$L(0) + R(0) + \frac{A}{C} = V$$

$$\frac{A}{C} = V$$

Solve for A.

$$A = CV$$

The particular solution is then

$$Q_p = CV,$$

and the general solution to the original ODE is

$$\begin{aligned}
Q(t) &= Q_c + Q_p \\
&= \exp\left(-\frac{R}{2L}t\right)\left(C_3\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t\right) + CV.
\end{aligned}$$

Differentiate it with respect to t.

$$\begin{aligned}\frac{dQ}{dt} = -\frac{R}{2L}\exp\left(-\frac{R}{2L}t\right)&\left(C_3\cos\frac{\sqrt{4LC-R^2C^2}}{2LC}t + C_4\sin\frac{\sqrt{4LC-R^2C^2}}{2LC}t\right)\\ &+ \exp\left(-\frac{R}{2L}t\right)\left(-C_3\frac{\sqrt{4LC-R^2C^2}}{2LC}\sin\frac{\sqrt{4LC-R^2C^2}}{2LC}t\right.\\ &\qquad\left.+ C_4\frac{\sqrt{4LC-R^2C^2}}{2LC}\cos\frac{\sqrt{4LC-R^2C^2}}{2LC}t\right)\end{aligned}$$

Apply the initial conditions to determine C_3 and C_4.

$$Q(0) = C_3 + CV = 0.01$$

$$\frac{dQ}{dt}(0) = -\frac{R}{2L}(C_3) + C_4\frac{\sqrt{4LC-R^2C^2}}{2LC} = 0$$

Solving this system yields

$$C_3 = \frac{1}{100} - CV \quad \text{and} \quad C_4 = \frac{RC\left(\frac{1}{100}-CV\right)}{\sqrt{4LC-R^2C^2}},$$

which means the solution to the initial value problem is

$$\begin{aligned}Q(t) &= \exp\left(-\frac{R}{2L}t\right)\left[\left(\frac{1}{100}-CV\right)\cos\frac{\sqrt{4LC-R^2C^2}}{2LC}t + \frac{RC\left(\frac{1}{100}-CV\right)}{\sqrt{4LC-R^2C^2}}\sin\frac{\sqrt{4LC-R^2C^2}}{2LC}t\right] + CV\\ &= \left(\frac{1}{100}-CV\right)\exp\left(-\frac{R}{2L}t\right)\left(\cos\frac{\sqrt{4LC-R^2C^2}}{2LC}t + \frac{RC}{\sqrt{4LC-R^2C^2}}\sin\frac{\sqrt{4LC-R^2C^2}}{2LC}t\right) + CV.\end{aligned}$$

Therefore, plugging in $L = 2$ H, $R = 40\ \Omega$, $C = 0.0025$ F, and $V = 12$ V,

$$Q(t) = -0.02e^{-10t}(\cos 10t + \sin 10t) + 0.03.$$

Below is a plot of $Q(t)$ (in Coulombs) versus t (in seconds).